Learning of the scientific approach at university : the case of research situations from problems of discrete mathematics

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Introduction

This poster aims to present a module of transversal and optional teaching proposed at the University of Grenoble about combinatorial games and mathematical reasonings. The primary objective is the learning of the scientific approach for non-specialist students in mathematics and in particular, the development of both reasonings and proving process. In this occasion, we propose an example of research situation built from a problem of discrete mathematics and proposed in this context. We will then show in what way this situation is a suitable candidate for our learning objectives highlighting some non-specialist students productions which will illustrate our purposes.

A module about combinatorial games and mathematical reasonings

A teaching module proposed at the University of Grenoble

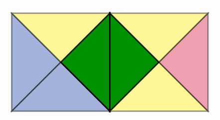
- For about twenty years, the teachers and researchers from the "Math à Modeler" team (http://mathsamodeler.ujf-grenoble.fr/) have been proposing a transversal and optional teaching concerning combinatorial games and reasoning mathematics.
- Open to non-specialist university students in mathematics of first and second year whose aim is the scientific approach learning (e.g., experimenting, questioning, conjecturing, reasoning and proving) based on problem-solving in mathematics.
- Research situations presented are from problems of discrete mathematics and built from a theoretical model about "**R**esearch-**S**ituations in **C**lassroom" denoted **RSC** (see, e.g., Grenier and Godot, 2004 ; Grenier and Payan, 2003 ; Ouvrier-Buffet, 2006).

Organization and assessment modalities about this teaching module

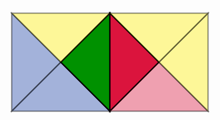
- The teaching module is open to first and second semester for the non-specialist university students in mathematics from first and second years (twenty-four hours per semester).
- Students work in group of three or four about a problem proposed during the first sessions (one to three sessions of two hours).
- Every pair of students must then choose a problem on which they are going to work for five to six weeks.
- The assessment focuses on a short dissertation concerning the chosen problem, followed by a presentation at the end of this module.

An example of RSC from Wang's problem to develop the scientific approach in mathematics

Wang's problem better known as the Domino Problem (Wang, 1961) is linked to some problematics about tilings of the whole plane or bounded discrete areas with a finite tileset. It has also been a precursor in many fields like first-order logic, computability and complexity for instance. More precisely, a Wang tile $\tau = (\tau_W, \tau_S, \tau_E, \tau_N)$ is a unit square with four colored edges. Two tiles can match if they have got a common edge of the same color. Rotations and reflections on the tiles are forbidden or else the tiling problem becomes obvious in $\mathbb{Z}^2$.



Valid pattern



Non-valid pattern

In 1966, Berger proved that Wang's problem was undecidable in the discrete plane $\mathbb{Z}^2$ and has been proven as being $\mathcal{NP}$-complete about a square or rectangle of size $p \times q$ (see, e.g., Lévy, 2011 ; Lewis, 1978). Let us note that it remains $\mathcal{NP}$-complete even when the rectangle sides are monochromes.

I- Analysis elements

From Wang's problem, we then elaborated a **RSC** which fulfils all the criteria of the **RSC** theoretical model (Da Ronch, Gandit and Gravier, 2020, pp. 99-101).

For our **RSC**, we consider a tileset $\mathcal{T}$ where each of the tiles has got exactly four different colours : blue, red, yellow and green for instance. Therefore, there are 4! permutations , thus 24 different tiles in $\mathcal{T}$. The decision problem coded $\mathcal{P}_0$ is so the following :

($\mathcal{P}_0$) Let $p > 0$, $q > 0$ two integers be, does the rectangle of size $p \times q$ whose edges are monochromes admit a valid tiling with some tiles of $\mathcal{T}$?

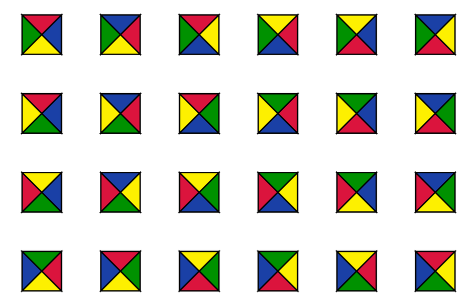


Fig. 1 *- Tileset $\mathcal{T}$ of 24 tiles.*

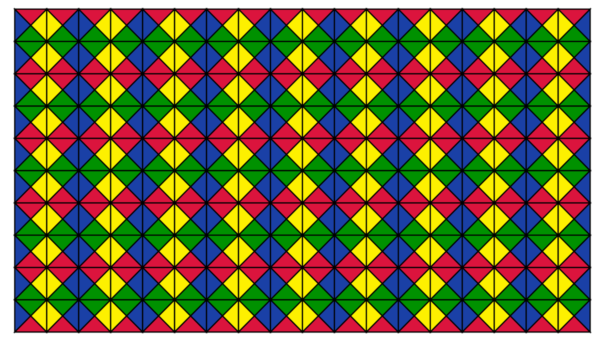


Fig. 2 *- Example of a valid tiling with some tiles of $\mathcal{T}$*

This problem led us to study three types of rectangles, denoted by A, B and C where the B-type has two sub-types $|B|$ and $\underline{\overline{B}}$ having respectively the same color on the east-west edges or the north-south edges (the choice of colors on the rectangle sides is arbitrary, Fig. 3). For the other types of rectangles which have at least two adjacent sides of the same color, it does not exist a tiling since no tile of $\mathcal{T}$ has twice the same color on a tile.

In Da Ronch, Gandit and Gravier (2020), we analyzed $\mathcal{P}_0$ from a mathematical and didactical point of view. In particular, we finely treated the C-type that we put in relationship with the other two types. We thus proved that for all $p \geqslant 1$ and $q \geqslant 1$, it always exists at least a valid tiling of a certain type. Moreover, to tile all the types of rectangle it is necessary and sufficient that $p > 1$ and $q > 1$ be simultaneously even (Tab. 1). Here, we will only develop the A-type in giving a few resolution strategies we put in relationship with some students' productions from short dissertions in order to show why this problem is a suitable candidate for the learning of the scientific approach.

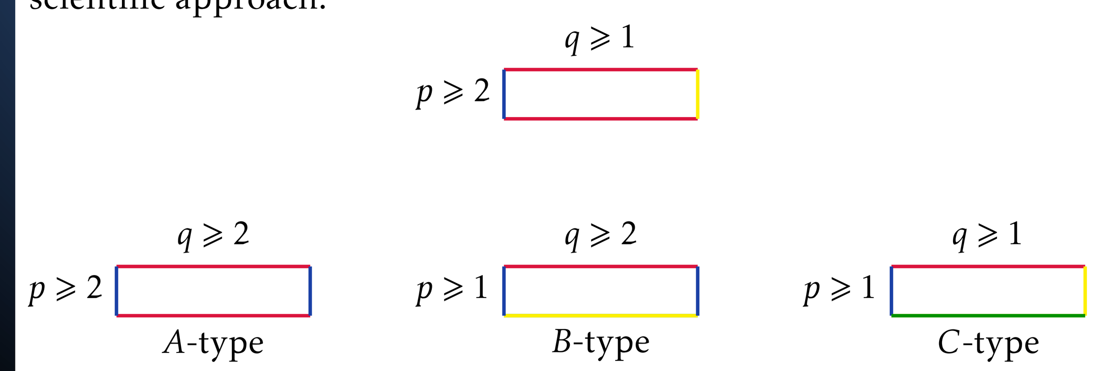


Fig. 3 *- Different types of rectangle.*

Type of rectangle	$p \in 2\mathbb{N}$ $q \in 2\mathbb{N}$	$p \in 2\mathbb{N}$ $q \notin 2\mathbb{N}$	$p \notin 2\mathbb{N}$ $q \in 2\mathbb{N}$	$p \notin 2\mathbb{N}$ $q \notin 2\mathbb{N}$
A-type	•	•	•	
$\lvert B\rvert$-type	•		•	
$\underline{\overline{B}}$-type	•	•		
C-type	•			•

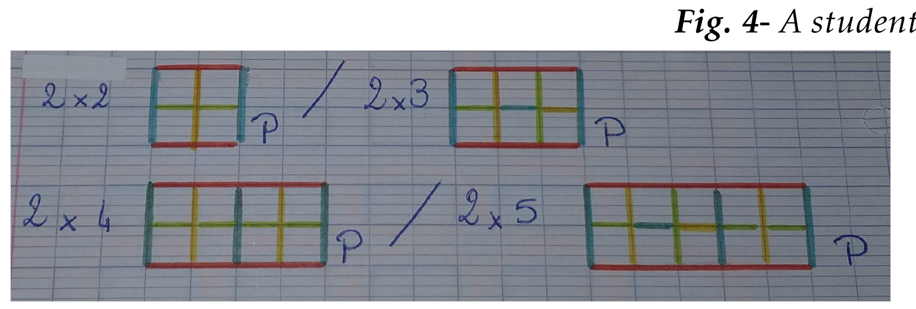
Tab. 1 *- Conditions about p and q to get a valid tiling.*

Let us consider the same problem as $\mathcal{P}_0$ but restricted to A-type, denoted $\mathcal{P}_1$. We can start to look for some conditions about $q \in \mathbb{N}^*$ so that the rectangles $1 \times q$, $2 \times q$ or $3 \times q$ can have a valid tiling.

- The rectangular tiling of size $1 \times q$ does not exist since there is no tile τ of $\mathcal{T}$ which has the same color to south and to north (i.e. $\tau_N = \tau_S$ does not exist in $\mathcal{T}$).

- For the $2 \times q$ rectangle of A-type, when $q \geqslant 2$ is even, it is sufficient to build a A-tiling of size 2×2 and to transfer horizontally $\frac{q}{2}$ times this pattern to tile the q-columns. When $q \geqslant 2$ is odd, it is sufficient to deconstruct the three previous last columns and to add a A-tiling of size 2×3. We thus showed that, for all the integers $p \geqslant 2$ and $q \geqslant 2$, it always exists a valid tiling for the $2 \times q$ rectangle of A-type. The students' production below puts forward this strategic about particular cases and the text beside explains the $2 \times q$ rectangle building from basic patterns (Fig. 4).

Fig. 4 *- A students' production proving the existence of a A-type tiling of size $2 \times q$.*



En testant les (2×c), on se rend compte que le (2×4) est composé de 2* (2×2).

De la même manière, le (2×5) est composé du (2×3) + (2×2).

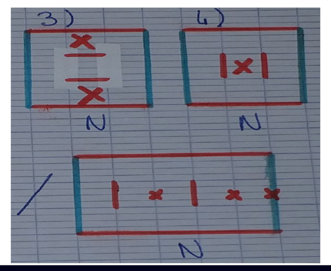
On en conclut alors que le (2×6) = 3* (2×2), le (2×7) = 2* (2×2)+ (2×3).

Par conséquent, les (2×c) sont tous pavables, comme une conjonctions de (2x2) et d'un (2x3) si nécessaire.

- For the $3 \times q$ rectangle of A-type, when $q \in 2\mathbb{N}^*$ the strategic is similary, it is sufficient to build a 3×2 rectangle of A-type and to transfer this pattern to tile the q-columns. Nevertheless, when q is odd the tiling does not exist. Let us assume this tiling exists, then the tiles of the middle row of this rectangle have necessarily an alternance of the red color since this color cannot be on the top and bottom of the middle row because of the fact there already is this color on the top and bottom of this rectangle. Thus, the tiling does not exist. Hereinafter a student's production which shows a strategic of forcage (proof by contradiction) about particular cases and which can extend to $3 \times q$ rectangle (Fig. 5). Let us notice all the same that this strategic of forcage is not efficient when $p > 3$ since there are too many sub-cases to deal with.

II- Analysis elements

Fig. 5 *- Another students' production proving a impossibility.*



On tente alors de comprendre : Il n'y a qu'une ligne au milieu, les lignes rouges ne peuvent donc pas être en haut ou en bas de cette ligne [3) et 4)], ainsi, le forçage rend le pavage impossible. En effet, par forçage, le carré se trouvant au centre du (3×3) ne se compose que de 2 couleurs.

Suite à de nombreux tests, nous avons conclu que le pavage est impossible sur un rectangle de taille (3×c) avec c impair. L'exemple du (3×5), ci-dessous le montre :

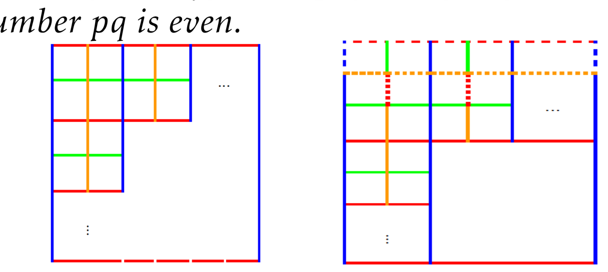
The study of these rectangles of the particular sizes generally leads students to formulate a general conjecture about the A-type, namely that the A-type rectangle is tileable, if and only if, the product of $p \geqslant 2$ by $q \geqslant 2$ is even, in other words this enables to give some conditions about the integers p and q to solve $\mathcal{P}_1$.

To prove the sufficient condition about the integers $p \geqslant 2$ and $q \geqslant 2$, we can use a similar strategic to the tilings of the $2 \times q$ rectangle from periodic patterns. One could also prove the existence of a valid tiling by using another established outcome about the rectangle of $|B|$-type and by adding, then, a $|B|$-type row (Fig. 6). Another strategic of proof would be to use the principle of infinite descent with the help of reasonings by induction, contradiction and minimal counterexample (an example is given in Da Ronch, Gandit and Gravier (2020) about the C-type). So, it allows to prove the existence of a $p \times q$ tiling of A-type when pq is even. However, the necessary condition is more subtil than the sufficient condition. Indeed, if the A-type is tileable then for each of the colors, the number of triangles $\mathcal{N}_\Delta$ from this color is equal to the sum of the number of the outside triangles $\mathcal{N}_{\Delta_{out}}$ and of the number of the inside triangles $\mathcal{N}_{\Delta_{in}}$ of this same color. Thus, we have

$$\mathcal{N}_\Delta = \mathcal{N}_{\Delta_{out}} + \mathcal{N}_{\Delta_{in}} = pq$$

In the case of the A-type, we necessarily have $\mathcal{N}_{\Delta_{in}} \in 2\mathbb{N}$ and $\mathcal{N}_{\Delta_{out}} \in 2\mathbb{N}$. Hence, it does not exist a tiling of A-type when p and q are odd. The established conjecture is then proved.

Fig. 6 *- A students' production proving the existence of a A-type tiling of size $p \times q$ when the number pq is even.*



Remark. This research situation has been analyzed in Da Ronch's PhD thesis in progress. In this occasion, we proved that regardless of the type of rectangle the problem is in fact solvable, if and only if, the tileset $\mathcal{T}$ has at least 5 colors. In other words, 5 is the number of colors necessary and sufficient within $\mathcal{T}$ to tile all the types of rectangle, and this, regardless of the integers $p > 1$ and $q > 1$.

Conclusion

We presented an example of the research situation proposed at the University of Grenoble in order to develop the non-specialist students' scientific approach in mathematics. In this occasion, we gave an example of the research situation arisen from Wang's problem and built from the Grenier's and Payan's RSC theoretical model whose details have been published in Da Ronch, Gandit and Gravier (2020).

On this point, we tried to show in what way this research situation is a suitable candidate for this learning objective in being based on a short mathematical analysis and some students' productions. In particular, we showed this situation can lead students to develop both inductive and deductive approaches bringing about the particular cases study, the conjecture formulation and the proof of it. This also enables to distinguish between sufficient and necessary conditions strongly bound to the existence or impossibility notions in mathematics.

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