The relationship between conceptual and procedural knowledge in calculus

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Background and Rationale

The study (Hechter, 2020) analyses relations between procedural and conceptual knowledge in order to guide optimal learning and inform teaching practices in calculus. The study is located within an augmented mathematics module for first year engineering students enrolled for an extended curriculum program in South Africa. The description of being 'conceptual' or 'procedural' is not necessarily a property of the task, but rather a description of the solution of the task (Engelbrecht, Bergsten, & Kagesten, 2009).

Research Questions

Primary research question:
What is the nature of conceptual and procedural knowledge in calculus?
Secondary research questions
1. **How can calculus items be analysed and classified with respect to conceptual and procedural knowledge?**
2. **How are conceptual and procedural constructs related in calculus?**

References

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Conceptual and procedural knowledge

Whether developing skills with symbols leads to conceptual understanding, or whether the presence of basic understanding should precede symbolic representation and skill practice, is one of the basic disagreements between the opposing sides of the so-called math wars. (Star 2005, p. 404)

Conceptual knowledge - knowledge of concepts or principles (Baroody, Feil, & Johnson, 2007; J.R. Star, 2005), "an integrated and functional grasp of mathematical ideas" (Kilpatrick et al., 2001, p. 118). **Procedural knowledge** is the ability to accurately and efficiently perform procedures to solve problems. **Mathematical proficiency requires both types** of mathematical knowledge.
Star (2005) states that conceptual and procedural knowledge refer not only to ***what*** is known (the type), but also to ***how*** the knowledge is known, therefore the quality of knowledge. The knowledge quality is described as deep or superficial (Star, 2005).

Problem solving strategies

Conceptual and procedural problem-solving categories	
C_1	**translations** between verbal, visual (graphical), numerical, and formal/algebraic mathematical expressions (representations)
$^1C_{2F}$	**linking** relationships wrt *functions*: functions ⟺ inverse functions, equation of a function
$^1C_{2D}$	**linking** relationships wrt *differentiation*: $f \Leftrightarrow f' \Leftrightarrow f'', D_{f'} \subseteq D_f$, $f'(x)=0 \Rightarrow f$ local extrema, $f'(x)>0 \Rightarrow f$ increasing, $f'(x)<0 \Rightarrow f$ decreasing, $f''(x)=0 \Rightarrow$ possible point of inflection, $f''(x)>0 \Rightarrow f$ concave up, $f''(x)<0 \Rightarrow f$ concave down, link position function (displacement) ⟹ velocity (speed) ⟹ acceleration
$^1C_{2I}$	**linking** relationships wrt *integration*: FTOC (part 1): $g'(x)=f(x)$, fix constant with integration techniques, link acceleration ⟹ velocity (speed) ⟹ position function (displacement) and total distance
$^1C_{3F}$	**interpretation of concepts** wrt *functions*: definitions, functions and relations, inverses, domain and range, restrictions, inequalities, concept of intersection and union
$^1C_{3D}$	**interpretation of concepts** wrt *differentiation*: gradient, continuity, differentiability, point of inflection, concavity
$^1C_{3I}$	**interpretation of concepts** wrt *integration*: FTOC, definite integral = enclosed net area
C_4	**applications** of concepts to mathematical situations (contextual problems)
P_1	symbolic and numerical **calculations**, substitution
$^1P_{2F}$	**rules** wrt *functions*, expressions (e.g. division by 0), equations (e.g. $a.b=0 \Rightarrow a=0$ or $b=0$), inequalities (e.g. division by -1), exponential laws (e.g. $a^0 = 1$), log laws, graph of the parabola, factorisation, $\sqrt{a^2}=\lvert a \rvert$
$^1P_{2D}$	*differentiation* **rules**
$^1P_{2I}$	*integration* **techniques**, FTOC (part 2)
P_3	**algorithms** (set of rules)
P_4	**formulae**
P_5	**symbols** (including notation)

Research design

- **Mixed-methods** approach, guided by **post positivism**
- **Qualitative** content analysis investigated conceptual and procedural knowledge relations ***within*** the solutions of 33 calculus items
  - the number of procedural and conceptual steps needed to answer the item (Engelbrecht, Bergsten, & Kågesten, 2012)
  - item label and item classification into one of four knowledge classes based on the type and quality of knowledge (Star, 2005).
- **Quantitative Rasch analysis**
  - analysed items were included in a data collection instrument administered to students
  - measure item difficulty and person proficiency
  - describe underlying cognitive construct ***between*** items
  - **Confirmatory factor analysis**

Knowledge classes (adapted Star, 2005)

Knowledge classes	number conceptual categories	number procedural categories
C^-P^-	2 or less	2 or less
C^+P^-	3 or more	2 or less
C^-P^+	2 or less	3 or more
C^+P^+	3 or more	3 or more

Example of content analysis

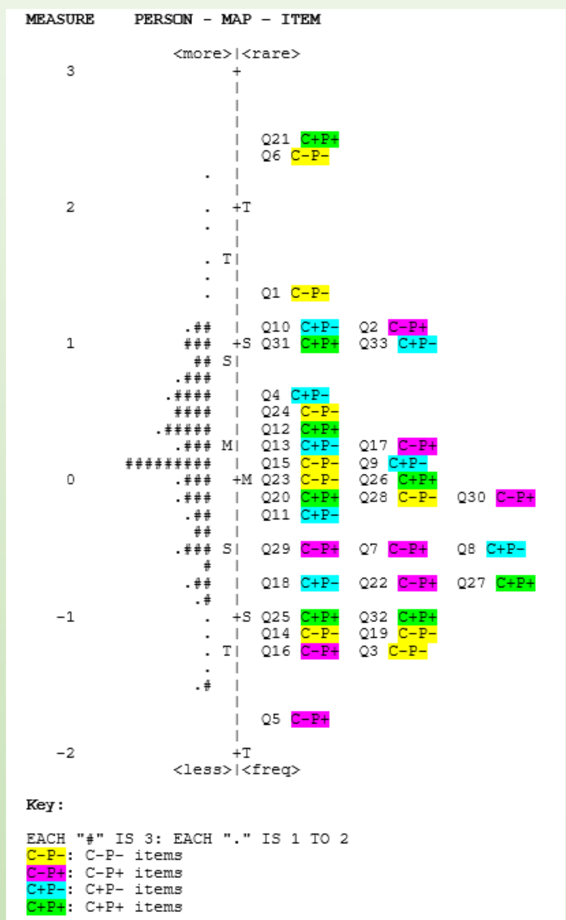
If a stone is thrown vertically upwards, the position function of the stone is given by
$s(t) = 30t - 5t^2 + 20$, where s is in metres and t is in seconds.

Calculate:
1. the time t when the stone will reach its maximum height
2. the maximum height of the stone (before it falls to the ground).

The expectation was that application of differentiation, more specific the concepts of the position function and velocity, would be used to solve the task. The analyses of approach 1 is seen below:

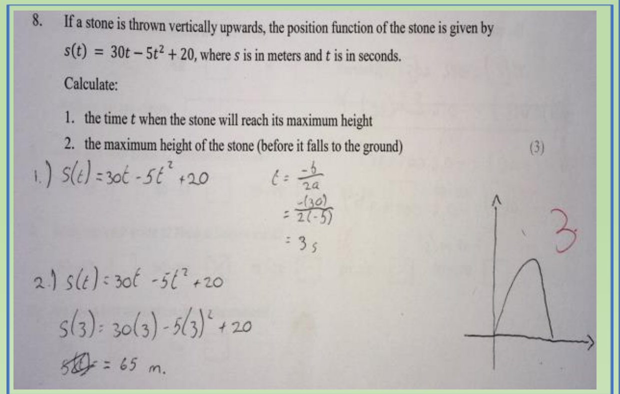
Approach 1		Results: C=5P=3
$s(t)=30t-5t^2+20$ (position function)	$C_4[1]$	$C_4[1]$: context - position function stone
$s'(t)=30-10t$ (velocity function)	P_{2D}	P_{2D}: differentiation rules
$s'(t)=0$ (velocity function = 0)	$C_4[2]$	$C_4[2]$: context - velocity function $\text{time}_{\text{velocity}}\ 0 \Rightarrow \text{time}_{\text{max height}}$
$\Rightarrow 30-10t=0$	$C_{2D}[1]$	$C_{2D}[1]$: link $f'(x)=0 \Rightarrow f$ local extrema
$\Rightarrow -10t=-30$ $\Rightarrow t=3s$	$P_1[1]$	$P_1[1]$: symbolic and numerical calculations
$s(3)=30t-5t^2+20$	$C_4[3]$	$C_4[3]$: context - position function max height $\Rightarrow \text{time}_{\text{velocity}}\ 0$
(position function at t = 3)	$C_{2D}[2]$	$C_{2D}[2]$: link position function (max height) and velocity (zero) at t = 3
$\Rightarrow s(3)=90-45+20$	$P_1[2]$	$P_1[2]$: substitution into position function
$\Rightarrow s(3)=65m$ (max height)	$P_1[1]$ $C_4[1]$	$P_1[1]$: numerical calculations $C_4[1]$: context - position function stone

Rasch Analysis: Person item map



The Rasch person–item map confirmed:
- items in the four classes were not clustered together per class
- item difficulty was not linked to the number of procedural and/or conceptual steps needed to do the mathematics

Different approach (student work)



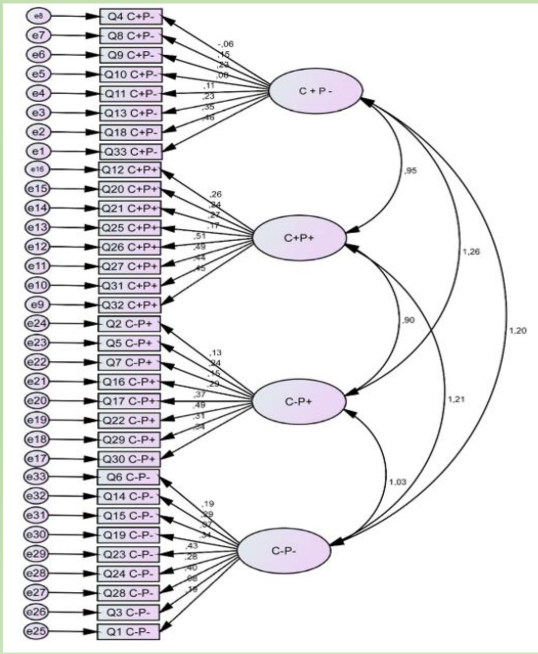
Findings

- item solutions drew on both procedural and conceptual components that cannot be separated
- item solutions could follow more than one approach
- analyses of the same item of solutions could differ, depending on the prior knowledge or experience since what is conceptual for one could be procedural for another
- item difficulty does not only depend on the number of procedural and conceptual steps

Conclusion

The study describes the relationship between the procedural and conceptual knowledge ***within*** and ***between*** items as integrated and complex (Kilpatrick et al., 2001).
The integrated relationship is visible on two levels:
- The **content analysis** reveals that the 33 calculus items follow conceptual and procedural steps to achieve problem-solving; however, the uniqueness of a problem-solving approach and the interpretation of an analysis are not absolute (Star & Stylianides, 2013).
- The **Rasch analysis** and person-item map confirmed the integrated relationship between procedural and conceptual knowledge, and indicated that the items had a range of difficulties and did not cluster together according to item label or class.
- **Suggestion:** teaching strategies should navigate between concepts and procedures, using different representations.

Confirmatory factor analysis



Confirmatory factor analysis indicated over-correlation between the classes:
- the defined classes cannot be separated
- **confirmed integration of procedural and conceptual cognitive processes**